

# Short proofs of strong normalization

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**Abstract.** This paper presents simple, syntactic strong normalization proofs for the simply-typed  $\lambda$ -calculus and the polymorphic  $\lambda$ -calculus (system **F**) with the full set of logical connectives, and all the permutative reductions. The normalization proofs use translations of terms and types of  $\lambda_{\rightarrow, \wedge, \vee, \perp}$  to terms and types of  $\lambda_{\rightarrow}$  and from  $\mathbf{F}_{\forall, \exists, \rightarrow, \wedge, \vee, \perp}$  to  $\mathbf{F}_{\forall, \rightarrow}$ .

## 1 Introduction

In this paper we consider the simply-typed and polymorphic lambda-calculus extended by type constructors corresponding to the usual logical connectives, namely conjunction, disjunction, absurdity and implication. In the polymorphic case we include both universal and existential quantification. In addition, we assume all the permutative conversions.

Different proofs of strong normalization of several variants of these calculi occur in the literature cf. [1,5,7,9,10]. It is however surprising that it is quite hard to find one covering the full set of connectives, applying to all the permutative conversions (in the polymorphic case none of the cited works does so) and given by a simple and straightforward argument. We can only repeat after J.Y. Girard: *I didn't find a proof really nice, and taking little space* [4, p. 130]. For instance, many proofs, like these in [7,9,10] are based on the computability method, or (in the polymorphic case) candidates of reducibility. This requires re-doing each time the same argument, but in a more complex way, due to the increased complexity of the language.

We believe that methodologically the most adequate approach is by reducing the question of strong normalization of the extended systems to the known strong normalization of the base systems, involving only implication and the universal quantifier. We propose two such proofs in what follows.

The first proof reduces the calculus  $\lambda_{\rightarrow, \wedge, \vee, \perp}$  with connectives  $\wedge, \vee, \rightarrow, \perp$  to the calculus  $\lambda_{\rightarrow}$ . Here we use the strong normalization of  $\lambda_{\rightarrow}$  with beta-eta-reductions. The proof is based on composing the ordinary reduction of classical connectives to implication and absurdity with Ong's translation of the  $\lambda\mu$ -

calculus to the ordinary  $\lambda\eta$ -calculus, as described e.g. in [8, Chapter 6]. To our knowledge this is the most direct way of showing SN for system  $\lambda_{\rightarrow, \wedge, \vee, \perp}$ .

The above method does not however extend to the polymorphic case. Indeed, the translation is strictly type-driven and requires an *a priori* knowledge of all types a given expression can obtain by polymorphic instantiation. Also the well known definition of logical connectives in system **F**:

$$\sigma \wedge \tau \equiv \forall t. (\sigma \rightarrow \tau \rightarrow t) \rightarrow t \quad \sigma \vee \tau \equiv \forall t. (\sigma \rightarrow t) \rightarrow (\tau \rightarrow t) \rightarrow t$$

is not adequate. The translation preserves beta-conversion, but not the permutations. The solution, first used by de Groote ([2], [3]), for first-order logic, is a CPS-translation. Our proof is similar to de Groote's but the version of CPS we use is based on Nakazawa and Tatsuta [6].

### 1.1 Definitions of relevant calculi

We consider the calculi  $\lambda_{\rightarrow, \wedge, \vee, \perp}$  and **F** <sub>$\forall, \exists, \rightarrow, \wedge, \vee, \perp$</sub>  in Church's style. The type  $\tau$  of a term  $M$  is written informally in upper index as  $M^\tau$ . However, if it is clear from the context, types will be omitted for the sake of brevity and readability – most right-hand sides of equations and reduction rules are written without types.

**The full simply-typed  $\lambda$ -calculus** Types of  $\lambda_{\rightarrow, \wedge, \vee, \perp}$  are built from multiple type constants; lowercase Greek letters are used to denote types.

**Definition 1.** Types of  $\lambda_{\rightarrow, \wedge, \vee, \perp}$

$$\sigma, \tau, \dots ::= p, q, \dots, \sigma \rightarrow \tau, \sigma \wedge \tau, \sigma \vee \tau, \perp$$

Syntax of terms of  $\lambda_{\rightarrow, \wedge, \vee, \perp}$  can be divided in two groups: constructor terms and eliminator terms. Lowercase Latin letters denote variables, uppercase – terms.

**Definition 2.** Terms of  $\lambda_{\rightarrow, \wedge, \vee, \perp}$

$$\begin{aligned} M, N, \dots &::= \text{Variables} \\ &\quad x^\sigma, y^\tau, \dots, \\ &\quad \text{Introduction} \\ &\quad (\lambda x^\sigma. N^\tau)^{\sigma \rightarrow \tau}, \langle M^\sigma, N^\tau \rangle^{\sigma \wedge \tau}, (\text{in}_1 A^\sigma)^{\sigma \vee \tau}, (\text{in}_2 B^\tau)^{\sigma \vee \tau} \\ &\quad \text{Elimination} \\ &\quad (M^{\sigma \rightarrow \tau} N^\sigma)^\tau, (P^{\sigma \wedge \tau} \pi_1)^\sigma, (P^{\sigma \wedge \tau} \pi_2)^\tau, (W^{\sigma \vee \tau} [x^\sigma. S^\delta, y^\tau. T^\delta])^\delta, \\ &\quad (A^\perp \epsilon_\tau)^\tau \end{aligned}$$

In the above, the notation  $\text{in}_1 A$  and  $\text{in}_2 A$  represents the left and right injection for the sum type,  $\pi_1$  and  $\pi_2$  are projections and  $W^{\sigma \vee \tau} [x. S^\delta, y. T^\delta]$  stands for a case statement. The epsilon represents the *ex falso*.

**Reductions** The beta-reductions are written as  $\rightarrow_\beta$  and commutative reductions are denoted by  $\rightsquigarrow$ . For any reduction  $\rightarrow$  transitive closure of this relation will be denoted as  $\rightarrow^+$  and transitive, reflexive closure as  $\rightarrow$ .

**Definition 3.**  $\beta$ -reductions in  $\lambda_{\rightarrow, \wedge, \vee, \perp}$

$$\begin{aligned} (\lambda x^\tau. M^\delta) A^\tau &\rightarrow_\beta M[x := A]^\delta \\ \langle M^\sigma, N^\tau \rangle \pi_1 &\rightarrow_\beta M^\sigma \\ \langle M^\sigma, N^\tau \rangle \pi_2 &\rightarrow_\beta N^\tau \\ (\text{in}_1 A)^{\sigma \vee \tau} [x^\sigma. S^\delta, y^\tau. T^\delta] &\rightarrow_\beta S[x^\sigma := A^\sigma]^\delta \\ (\text{in}_2 B)^{\sigma \vee \tau} [x^\sigma. S^\delta, y^\tau. T^\delta] &\rightarrow_\beta S[y^\tau := B^\tau]^\delta \end{aligned}$$

**Definition 4.** Commutative reductions in  $\lambda_{\rightarrow, \wedge, \vee, \perp}$

$$\begin{aligned} (A^\perp \epsilon_{\sigma \rightarrow \tau}) N^\sigma &\rightsquigarrow A^\perp \epsilon_\tau \\ (A^\perp \epsilon_{\sigma \wedge \tau}) \pi_1 &\rightsquigarrow A^\perp \epsilon_\sigma \\ (A^\perp \epsilon_{\sigma \wedge \tau}) \pi_2 &\rightsquigarrow A^\perp \epsilon_\tau \\ (A^\perp \epsilon_{\sigma \vee \tau}) [x^\sigma. S^\delta, y^\tau. T^\delta] &\rightsquigarrow A^\perp \epsilon_\delta \\ (A^\perp \epsilon_\perp) \epsilon_\sigma &\rightsquigarrow A^\perp \epsilon_\sigma \\ ((W^{\sigma \vee \tau} [x. S^{\alpha \rightarrow \beta}, y. T^{\alpha \rightarrow \beta}]) N^\alpha)^\beta &\rightsquigarrow W^{\sigma \vee \tau} [x. (SN)^\beta, y. (TN)^\beta] \\ ((W^{\sigma \vee \tau} [x. S^{\alpha \wedge \beta}, y. T^{\alpha \wedge \beta}]) \pi_1)^\alpha &\rightsquigarrow W^{\sigma \vee \tau} [x. (S\pi_1)^\alpha, y. (T\pi_1)^\alpha] \\ ((W^{\sigma \vee \tau} [x. S^{\alpha \wedge \beta}, y. T^{\alpha \wedge \beta}]) \pi_2)^\beta &\rightsquigarrow W^{\sigma \vee \tau} [x. (S\pi_2)^\beta, y. (T\pi_2)^\beta] \\ (W^{\sigma \vee \tau} [x. S^{\alpha \vee \beta}, y. T^{\alpha \vee \beta}]) [a^\alpha. A^\delta, b^\beta. B^\delta] &\rightsquigarrow \\ W^{\sigma \vee \tau} [x. S[a. A^\delta, b. B^\delta], y. T[a. A^\delta, b. B^\delta]] & \\ (W^{\sigma \vee \tau} [x. S^\perp, y. T^\perp]) \epsilon_\alpha &\rightsquigarrow W^{\sigma \vee \tau} [x. S\epsilon_\alpha, y. T\epsilon_\alpha] \end{aligned}$$

Note that the above commutative reductions follow these two patterns:

$$(W[x.S, y.T])E \rightsquigarrow W[x.SE, y.TE], \quad (1)$$

$$(A\epsilon)E \rightsquigarrow A\epsilon, \quad (2)$$

where  $E$  is an arbitrary eliminator. That is,  $E$  is either a term  $N$  or a projection, or epsilon, or it has the form  $[x.S, y.T]$ .

**The full polymorphic  $\lambda$ -calculus** The full polymorphic  $\lambda$ -calculus extends the system of the previous section by existential and universal polymorphism. Terms of the calculus are all the terms of simply-typed  $\lambda$  calculus plus universal and existential introduction and elimination.

**Definition 5.** Types of  $\mathbf{F}_{\vee, \exists, \rightarrow, \wedge, \vee, \perp}$

$$\sigma, \tau, \dots ::= p, q, \dots, \sigma \rightarrow \tau, \sigma \wedge \tau, \sigma \vee \tau, \forall p \tau, \exists p \tau, \perp$$

In the definition below, notation  $[M^{\tau[p:=\sigma]}, \sigma]$  stands for introduction of type  $\exists p \tau$  and  $[x^\tau.N^\delta]$  is an eliminator for that type.

**Definition 6.** Terms of  $\mathbf{F}_{\forall, \exists, \rightarrow, \wedge, \vee, \perp}$

$$\begin{aligned}
& M, N, \dots ::= \text{Variables} \\
& \quad x^\sigma, y^\tau, \dots \\
& \quad \text{Introductions} \\
& \quad (\lambda x^\sigma.N^\tau)^{\sigma \rightarrow \tau}, \langle M^\sigma, N^\tau \rangle^{\sigma \wedge \tau}, (\mathbf{in}_1 A^\sigma)^{\sigma \vee \tau}, (\mathbf{in}_2 B^\tau)^{\sigma \vee \tau}, \\
& \quad [M^{\tau[p:=\sigma]}, \sigma]^{\exists p \tau}, (\Lambda p M^\tau)^{\forall p \tau} \\
& \quad \text{Eliminations} \\
& \quad (M^{\sigma \rightarrow \tau} N^\tau)^\tau, (P^{\sigma \wedge \tau} \pi_1)^\sigma, (P^{\sigma \wedge \tau} \pi_2)^\tau, (W^{\sigma \vee \tau} [x^\sigma.S^\delta, y^\tau.T^\delta])^\delta, \\
& \quad (M^{\exists p \tau} [x^\tau.N^\delta])^\delta, (M^{\forall p \tau} \sigma)^\tau [p:=\sigma] \\
& \quad (A^\perp \epsilon_\tau)^\tau
\end{aligned}$$

The  $\beta$ -reductions and commutative reductions in this system are as follows.

**Definition 7.** The  $\beta$ -reductions in  $\mathbf{F}_{\forall, \exists, \rightarrow, \wedge, \vee, \perp}$  are as in Definition 3 and in addition

$$[M^{\tau[p:=\sigma]}, \sigma][x^\tau.N^\delta] \rightarrow_\beta (N[p := \sigma][x := M])^\delta \quad (3)$$

$$(\Lambda p M^\tau)\sigma \rightarrow_\beta M[p := \sigma] \quad (4)$$

The total number of commutative reductions reaches 21. The patterns mentioned in Rules (1) and (2) are extended by the additional one:

$$(M[x.P])E \rightsquigarrow M[x.PE], \quad (5)$$

where  $E$  can also be of the form of existential  $([y.R])$  or universal  $(\sigma)$  eliminator.

**Definition 8.** Additional commutative reductions in  $\mathbf{F}_{\forall, \exists, \rightarrow, \wedge, \vee, \perp}$ .

Let  $\delta$  abbreviate  $\forall p \alpha$  in rules below.

$$(W^{\sigma \vee \tau} [x^\sigma.S^\delta, y^\tau.T^\delta])\gamma \rightsquigarrow W[x.(S\gamma)^{\alpha[p:=\gamma]}, y.(T\gamma)^{\alpha[p:=\gamma]}] \quad (6)$$

$$(A^\perp \epsilon_\delta)\gamma \rightsquigarrow A^\perp \epsilon_{\alpha[p:=\gamma]} \quad (7)$$

$$(M^{\exists p \tau} [x^\tau.P^\delta])\gamma \rightsquigarrow M^{\exists p \tau} [x.(P\gamma)^{\alpha[p:=\gamma]}] \quad (8)$$

$$(9)$$

In the following rules,  $\delta$  abbreviates  $\exists p \alpha$ .

$$(W^{\sigma \vee \tau} [x^\sigma.S^\delta, y^\tau.T^\delta])[a^\alpha.N^\xi] \rightsquigarrow W^{\sigma \vee \tau} [x.(S[a.N])^\xi, y.(T[a.N])^\xi] \quad (10)$$

$$(A^\perp \epsilon_\delta)[a^\alpha.N^\xi] \rightsquigarrow A^\perp \epsilon_\xi \quad (11)$$

$$(M^{\exists p \tau} [y^\tau.P^\delta])[a^\alpha.N^\xi] \rightsquigarrow M^{\exists p \tau} [y.(P[a.N])^\xi] \quad (12)$$

$$A^\delta[x^\alpha.N^{\sigma \rightarrow \tau}]P^\sigma \rightsquigarrow A[x.(NP)^\tau] \quad (13)$$

$$A^\delta[x^\alpha.N^{\sigma \wedge \tau}]\pi_1 \rightsquigarrow A[x.(N\pi_1)^\sigma] \quad (14)$$

$$A^\delta[x^\alpha.N^{\sigma \wedge \tau}]\pi_2 \rightsquigarrow A[x.(N\pi_2)^\tau] \quad (15)$$

$$A^\delta[x^\alpha.N^{\sigma \vee \tau}][y^\sigma.S^\delta, z^\tau.T^\delta] \rightsquigarrow A[x.(N[y.S, z.T])^\delta] \quad (16)$$

$$A^\delta[x^\alpha.N^\perp]\epsilon_\sigma \rightsquigarrow A[x.(N\epsilon_\sigma)^\sigma] \quad (17)$$

## 2 The translation for simple types

A type  $\tau$  of the  $\lambda_{\rightarrow, \wedge, \vee, \perp}$  calculus is translated to a type  $|\tau|$  of  $\lambda_{\rightarrow}$  calculus, a term  $M$  is translated to a term  $|M|$ .

**Definition 9.** Translation of types.

$$\begin{aligned} |\alpha| &= \perp, \text{ for all type constants } \alpha = \perp, p, q, \dots \\ |\sigma \rightarrow \tau| &= |\sigma| \rightarrow |\tau| \\ |\sigma \wedge \tau| &= (|\sigma| \rightarrow |\tau| \rightarrow \perp) \rightarrow \perp \\ |\sigma \vee \tau| &= (|\sigma| \rightarrow \perp) \rightarrow (|\tau| \rightarrow \perp) \rightarrow \perp \end{aligned}$$

*Example 10.* Let  $\tau = p \rightarrow q \rightarrow (p \wedge q)$ . Then

$$|\tau| = \perp \rightarrow \perp \rightarrow (\perp \rightarrow \perp \rightarrow \perp) \rightarrow \perp.$$

**Definition 11.** (Translation of terms) It is assumed below that types  $|\sigma|, |\tau|$  and  $|\delta|$  are as follows:  $|\sigma| = \sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow \perp$ ,  $|\tau| = \tau_1 \rightarrow \dots \rightarrow \tau_m \rightarrow \perp$  and  $|\delta| = \delta_1 \rightarrow \dots \rightarrow \delta_k \rightarrow \perp$ .

$$|x^\sigma| = x^{|\sigma|} \quad (18)$$

$$|\lambda x^\tau.M^\sigma| = \lambda x^{|\tau|}.|M|^{|\sigma|} \quad (19)$$

$$|(M, N)^{\sigma \wedge \tau}| = \lambda z^{|\sigma| \rightarrow |\tau| \rightarrow \perp}.z|M|^{|\sigma|}|N|^{|\tau|} \quad (20)$$

$$|(\text{in}_1 A)^{\sigma \vee \tau}| = \lambda x^{|\sigma| \rightarrow \perp}.\lambda y^{|\tau| \rightarrow \perp}.x|A|^{|\sigma|} \quad (21)$$

$$|(\text{in}_2 B)^{\sigma \vee \tau}| = \lambda x^{|\sigma| \rightarrow \perp}.\lambda y^{|\tau| \rightarrow \perp}.x|B|^{|\tau|} \quad (22)$$

$$|(M^{\sigma \rightarrow \tau} N^\sigma)| = (|M|^{|\sigma| \rightarrow |\tau|}|N|^{|\sigma|}) \quad (23)$$

$$\begin{aligned} |(P^{\sigma \wedge \tau})\pi_1| &= \lambda x_1^{\sigma_1} \dots \lambda x_n^{\sigma_n}.|P|^{|\sigma \wedge \tau|} \\ &\quad (\lambda x^{|\sigma|}.\lambda y^{|\tau|}.(xx_1 \dots x_n)^\perp) \end{aligned} \quad (24)$$

$$\begin{aligned} |(P^{\sigma \wedge \tau})\pi_2| &= \lambda x_1^{\tau_1} \dots \lambda x_m^{\tau_m}.|P|^{|\sigma \wedge \tau|} \\ &\quad (\lambda x^{|\sigma|}.\lambda y^{|\tau|}.(yx_1 \dots x_m)^\perp) \end{aligned} \quad (25)$$

$$\begin{aligned} |A^{\sigma \vee \tau}[x.S^\delta, y.T^\delta]| &= \lambda x_1^{\delta_1} \dots \lambda x_k^{\delta_k}.|A|^{(|\sigma| \rightarrow \perp) \rightarrow (|\tau| \rightarrow \perp) \rightarrow \perp} \\ &\quad (\lambda x^{|\sigma|}.\lambda y^{|\delta|}x_1 \dots x_k)(\lambda y^{|\tau|}.\lambda T^{|\delta|}x_1 \dots x_k) \end{aligned} \quad (26)$$

$$|M^\perp \epsilon_\sigma| = \lambda x_1^{\sigma_1} \dots \lambda x_{n-1}^{\sigma_{n-1}}.|M|^\perp \quad (27)$$

**Lemma 12 (Soundness).** *If a term  $M$  has type  $\delta$ , then  $|M|$  has type  $|\delta|$ .*

*Proof.* Obvious. □

**Lemma 13.** *If  $R \rightarrow R'$ , then  $|R| \rightarrow_{\beta\eta}^+ |R'|$ .*

*Proof.* The proof proceeds by cases on the definition of  $\rightarrow_\beta$  and  $\rightsquigarrow$ . Two example reductions will be elaborated here.

(24) Let  $R = \langle M^\sigma, N^\tau \rangle \pi_1$  and  $R \rightarrow_\beta R' = M$ , where  $|\sigma| = \sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow \perp$ .

$$\begin{aligned}
|R| &= |\langle M, N \rangle^{\sigma \wedge \tau} \pi_1| \\
&= \lambda a_1^{\sigma_1} \dots \lambda a_n^{\sigma_n} . |\langle M, N \rangle|^{|\sigma \wedge \tau|} (\lambda x^{|\sigma|} . \lambda y^{|\tau|} . (x a_1 \dots a_n)^\perp) \\
&= \lambda \vec{a} . (\lambda z^{|\sigma| \rightarrow |\tau| \rightarrow \perp} . z |M| |N|) (\lambda x^{|\sigma|} \lambda y^{|\tau|} . (x \vec{a})^\perp) \\
&\rightarrow_\beta \lambda \vec{a} . ((\lambda x^{|\sigma|} \lambda y^{|\tau|} . (x \vec{a})^\perp) |M| |N|) \\
&\rightarrow_\beta \lambda \vec{a} . (\lambda y^{|\tau|} . |M| \vec{a}) |N| \rightarrow_\beta \lambda \vec{a} . |M| \vec{a} \rightarrow_\eta^+ |M| \\
&= |R'|
\end{aligned}$$

(26) Let  $R = (W^{\sigma \vee \tau} [x . S^{\alpha \rightarrow \beta}, y . T^{\alpha \rightarrow \beta}]) N^\alpha$  and let  $R' = W^{\sigma \vee \tau} [x . (SN)^\beta, y . (TN)^\beta]$ . Then  $R \rightsquigarrow R'$ , according to (26). Assuming  $|\beta| = \beta_1 \rightarrow \dots \rightarrow \beta_n \rightarrow \perp$ , we have

$$\begin{aligned}
|R| &= (\lambda a^{|\alpha|} b_1^{\beta_1} \dots b_n^{\beta_n} . |W| (\lambda x^{|\sigma|} . |S|^{|\alpha| \rightarrow |\beta|} \vec{a} \vec{b}) (\lambda y^{|\tau|} . |T|^{|\alpha| \rightarrow |\beta|} \vec{a} \vec{b})) |N|^{|\alpha|} \\
&\rightarrow_\beta \lambda b_1 \dots b_n . |W| (\lambda x^{|\sigma|} . |S| |N| \vec{b}) (\lambda y^{|\tau|} . |T| |N| \vec{b}) \\
&= |R'|
\end{aligned}$$

Other cases are similar. □

**Theorem 14.** *The calculus  $\lambda_{\rightarrow, \wedge, \vee, \perp}$  is strongly normalizing.*

*Proof.* Suppose, by contradiction, that  $M^\tau$  admits an infinite  $\beta$ -reduction

$$M^\tau = M_0^\tau \rightarrow_\beta M_1^\tau \rightarrow_\beta M_2^\tau \rightarrow_\beta \dots$$

By Theorem 13 we have an infinite reduction in  $\lambda_{\rightarrow}$

$$|M^\tau| = |M_0| \rightarrow_{\beta\eta}^+ |M_1| \rightarrow_{\beta\eta}^+ |M_2| \rightarrow_{\beta\eta}^+ \dots$$

This contradicts the SN property of  $\lambda_{\rightarrow}$ . □

### 3 Translation for polymorphic types

As we mentioned in the introduction, the translations in Section 3 are not adequate for the polymorphic case and therefore we apply a call-by-name CPS translation. In general, a type  $\tau$  is translated to  $\underline{\tau} = (\tau^* \rightarrow \perp) \rightarrow \perp$ . This translation, unlike the one for simple types, does not unify type constants. The helper translation  $*$  is given below.

**Definition 15.** Helper translation  $*$ .

$$\begin{aligned}
\alpha^* &= \alpha, \text{ for all type constants } \alpha = \perp, p, q, \dots \\
(\alpha \rightarrow \beta)^* &= \underline{\alpha} \rightarrow \underline{\beta} \\
(\alpha \wedge \beta)^* &= (\underline{\alpha} \rightarrow \underline{\beta} \rightarrow \perp) \rightarrow \perp \\
(\alpha \vee \beta)^* &= (\underline{\alpha} \rightarrow \perp) \rightarrow (\underline{\beta} \rightarrow \perp) \rightarrow \perp \\
(\forall p \tau)^* &= \forall p \underline{\tau} \\
(\exists p \tau)^* &= (\forall p (\underline{\tau} \rightarrow \perp)) \rightarrow \perp
\end{aligned}$$

A term  $M^\tau$  is translated to the term  $\underline{M} = \lambda k^{\tau^* \rightarrow \perp}. (M \diamond k)$ . To achieve that, two helper translations are needed:  $\diamond$  and  $@$ . The term  $K$  in the definition below is of type  $\tau^* \rightarrow \perp$ . The term  $M \diamond K$  is always of type  $\perp$ .

**Definition 16.** Helper translation  $\diamond$

$$x^\tau \diamond K = xK \quad (28)$$

$$\lambda x^\sigma. N^\rho \diamond K = K(\lambda x^\sigma. \underline{N}) \quad (29)$$

$$\langle N_1^{\tau_1}, N_2^{\tau_2} \rangle \diamond K = K(\lambda p^{\tau_1 \rightarrow \tau_2 \rightarrow \perp}. p \underline{N_1} \underline{N_2}) \quad (30)$$

$$(\text{in}_1 A)^{\tau_1 \vee \tau_2} \diamond K = K(\lambda a^{\tau_1 \rightarrow \perp} b^{\tau_2 \rightarrow \perp}. a \underline{A}) \quad (31)$$

$$(\text{in}_2 B)^{\tau_1 \vee \tau_2} \diamond K = K(\lambda a^{\tau_1 \rightarrow \perp} b^{\tau_2 \rightarrow \perp}. b \underline{B}) \quad (32)$$

$$\Lambda p N^\rho \diamond K = K(\Lambda p. \underline{N}) \quad (33)$$

$$[N^{\rho[p:=\sigma]}, \sigma] \diamond K = K(\lambda u^{\forall p(\rho \rightarrow \perp)}. u \underline{\sigma} \underline{N}) \quad (34)$$

$$NE \diamond K = N \diamond (E @ K) \quad (35)$$

In (35) the symbol  $E$  stands for an arbitrary eliminator. That is,  $E$  is one of the expressions  $\{R^\sigma, \pi_1, \pi_2, [x^{\tau_1}. S^\delta, y^{\tau_2}. T^\delta], \sigma, [x^\rho. S^\delta], \epsilon_\alpha\}$  and the omitted type of term  $N$  is appropriate for every eliminator  $E$ .

**Definition 17.** Helper translation  $@$

$$R @ K = \lambda m^{\sigma \rightarrow \perp}. m \underline{R} K$$

$$\pi_1 @ K = \lambda m^{(\tau_1 \rightarrow \tau_2 \rightarrow \perp) \rightarrow \perp}. m(\lambda a^{\tau_1} b^{\tau_2}. a K)$$

$$\pi_2 @ K = \lambda m^{(\tau_1 \rightarrow \tau_2 \rightarrow \perp) \rightarrow \perp}. m(\lambda a^{\tau_1} b^{\tau_2}. b K)$$

$$[x^{\tau_1}. S^\delta, y^{\tau_2}. T^\delta] @ K = \lambda m^{(\tau_1 \rightarrow \perp) \rightarrow (\tau_2 \rightarrow \perp) \rightarrow \perp}.$$

$$m(\lambda x^{\tau_1}. (S \diamond K))(\lambda y^{\tau_2}. (T \diamond K))$$

$$\sigma @ K = \lambda m^{\forall p \rho}. m \underline{\sigma} K$$

$$[x^\rho. S^\delta] @ K = \lambda m^{(\forall p(\rho \rightarrow \perp)) \rightarrow \perp}. m(\Lambda p \lambda x^\rho. (S \diamond K))$$

$$\epsilon_\alpha @ K = \lambda m^\perp. m$$

**Lemma 18.** [Soundness] *If a term  $M$  has type  $\delta$ , then  $\underline{M}$  has type  $\underline{\delta}$ .*

*Proof.* Easy. □

**Lemma 19.** [Properties of substitution] *For a term  $R$  and any term  $K$  and for any types  $\tau$  and  $\rho$  the following holds:*

$$\underline{R}[x^\delta := \underline{N}^\delta] =_\alpha \underline{R}[x := N]; \quad (36)$$

$$(R \diamond K)[x^\delta := \underline{N}^\delta] =_\alpha R[x := N] \diamond K[x := N]; \quad (37)$$

$$(R @ K)[x^\delta := \underline{N}^\delta] =_\alpha R[x := N] @ K[x := N] \text{ if } R \text{ is an eliminator}; \quad (38)$$

$$\underline{\tau}[p := \underline{\rho}] =_\alpha \underline{\tau}[p := \rho]; \quad (39)$$

$$(R \diamond K)[p := \underline{\rho}] =_\alpha R[p := \rho] \diamond K[p := \rho]; \quad (40)$$

$$(R @ K)[p := \underline{\rho}] =_\alpha R[p := \rho] @ K[p := \rho] \text{ if } R \text{ is an eliminator}. \quad (41)$$

*Proof.* This lemma is proved by simultaneous induction on the definition of substitution.  $\square$

**Lemma 20.** *If  $R \rightarrow_\beta R'$ , then  $\underline{R} \rightarrow_\beta^+ \underline{R}'$ .*

*Proof.* Using induction on the definition of  $\rightarrow_\beta$  we have 7 cases. For example, consider (3), where  $R = [M^{\tau[p:=\sigma]}, \sigma][x^\tau.N^\delta]$  and  $R' = (N[p := \sigma][x := M])^\delta$ .

$$\begin{aligned} (3) \quad \underline{R} &= \lambda k. (\lambda m^{(\exists p \tau)^*}. m(\Lambda p \lambda x^\tau. (N \diamond k))) (\lambda u^{\forall p (\exists \rightarrow \perp)}. u \underline{\sigma} \underline{M}) \\ &\rightarrow_\beta \lambda k. (\lambda u. u \underline{\sigma} \underline{M}) (\Lambda p \lambda x. (N \diamond k)) \\ &\rightarrow_\beta \lambda k. (\Lambda p \lambda x. (N \diamond k)) \underline{\sigma} \underline{M} \\ &\rightarrow_\beta \lambda k. (\lambda x. (N \diamond k)) [p := \underline{\sigma}] \underline{M} \\ &\rightarrow_\beta \lambda k. (\lambda x. (N[p := \sigma] \diamond k)) \underline{M} \quad (\text{from (40)}) \\ &\rightarrow_\beta \lambda k. (N[p := \sigma] \diamond k) [x := \underline{M}] \\ &=_\alpha \lambda k. (N[p := \sigma][x := M] \diamond k) \quad (\text{from (37)}) \\ &= \underline{R}' \end{aligned}$$

$\square$

**Lemma 21.** *If  $R \rightsquigarrow R'$ , then  $\underline{R} =_\alpha \underline{R}'$ .*

*Proof.* The complete proof consists of 21 cases. Here, two interesting commutations will be elaborated. The other cases are similar and left to the reader.

From (12) we get

$$\begin{aligned} \underline{\text{LHS}} &= \lambda k. (M[y.P] \diamond ([x.N] @ k)) = \lambda k. (M \diamond ([y.P] @ ([x.N] @ k))) \\ &= \lambda k. (M \diamond (\lambda m. m(\Lambda p \lambda y. (P \diamond [x.N] @ k)))) \\ \underline{\text{RHS}} &= \lambda k. (M \diamond ([y.P[x.N]] @ k)) = \lambda k. (M \diamond (\lambda m. m(\Lambda p \lambda y. (P[x.N] \diamond k)))) \\ &= \lambda k. (M \diamond (\lambda m. m(\Lambda p \lambda y. (P \diamond [x.N] @ k)))) \end{aligned}$$



From (17) we get

$$\begin{aligned}
\text{LHS} &= \lambda k. (A[x.N] \diamond (\epsilon_\sigma @ k)) = \lambda k. (A[x.N] \diamond (\epsilon_\sigma @ k)) \\
&= \lambda k. (A \diamond ([x.N] @ (\epsilon_\sigma @ k))) \\
&= \lambda k. (A \diamond (\lambda m. m(\Lambda p \lambda x. (N \diamond (\epsilon_\sigma @ k)))))) \\
\text{RHS} &= \lambda k. (A \diamond ([x.N \epsilon_\sigma] @ k)) = \lambda k. (A \diamond (\lambda m. m(\Lambda p \lambda x. (N \epsilon_\sigma \diamond k)))) \\
&= \lambda k. (A \diamond (\lambda m. m(\Lambda p \lambda x. (N \diamond (\epsilon_\sigma @ k))))))
\end{aligned}$$

□

**Lemma 22.** *Every sequence of commutative reductions in  $\mathbf{F}_{\forall, \exists, \rightarrow, \wedge, \vee, \perp}$  must terminate.*

*Proof.* To prove this lemma we define such a measure  $\chi(M) > 0$ , that for any commutation  $M \rightsquigarrow M'$ , we have  $\chi(M) > \chi(M')$ . Please note, that we have 3 patterns of commutative reductions in Rules (1), (2) and (5). We use those patterns to define appropriate conditions for measure  $\chi$ :

$$\chi((W[x.S, y.T])E) > \chi(W[x.SE, y.TE]) \quad (42)$$

$$\chi((A\epsilon)E) > \chi(A\epsilon) \quad (43)$$

$$\chi((N[x.P])E) > \chi(N[x.PE]) \quad (44)$$

$$\chi(M) \geq 1$$

Now we give the definition of the function  $\chi(M)$ ; it is similar to de Groote's norm  $|\cdot|$  from [2] but simpler:

$$\begin{aligned}
\chi(x) &= 1 \\
\chi(\lambda x. N) &= \chi(\text{in}_1 N) = \chi(\text{in}_2 N) = \chi(N), \quad \chi(\langle M_1, M_2 \rangle) = \chi(M_1) + \chi(M_2) \\
\chi(FA) &= \chi(F)^2 \chi(A), \quad \chi(P\pi_1) = \chi(P\pi_2) = \chi(P)^2, \quad \chi(N\sigma) = \chi(N)^2 \\
\chi(W[x.S, y.T]) &= \chi(W)^2 (\chi(S) + \chi(T)) + 1 \quad \chi(N[x.P]) = \chi(N)^2 \chi(P) + 1 \\
\chi(A\epsilon) &= \chi(A)^2 + 1
\end{aligned}$$

There are 21 easy cases, one for each permutation from Definitions 4 and 8. We will show here one example case for each pattern mentioned above.

$$(42) \text{ Let } l = \chi((W[x.S, y.T])[a.A, b.B]) \text{ and } r = \chi(W[x.S[a.A, b.B], y.T[a.A, b.B]]).$$

$$\begin{aligned}
l &= \chi(W[x.S, y.T])^2 (\chi(A) + \chi(B)) + 1 \\
&= (\chi(W)^2 (\chi(S) + \chi(T)) + 1)^2 (\chi(A) + \chi(B)) + 1 \\
&> (\chi(W)^2 (\chi(S) + \chi(T)))^2 (\chi(A) + \chi(B)) + 1 \\
&= \chi(W)^4 ((\chi(S)^2 + \chi(T)^2)(\chi(A) + \chi(B)) + 2(\chi(S)\chi(T))(\chi(A) + \chi(B))) + 1 \\
&> \chi(W)^4 ((\chi(S)^2 + \chi(T)^2)(\chi(A) + \chi(B)) + 2) + 1
\end{aligned}$$

$$\begin{aligned}
r &= \chi(W)^2(\chi(S[a.A, b.B]) + \chi(T[a.A, b.B])) + 1 \\
&= \chi(W)^2(\chi(S)^2(\chi(A) + \chi(B)) + 1 + \chi(T)^2(\chi(A) + \chi(B)) + 1) + 1 \\
&= \chi(W)^2((\chi(S)^2 + \chi(T)^2)(\chi(A) + \chi(B)) + 2) + 1 \\
l &> r
\end{aligned}$$

(43) Let  $l = \chi((A\epsilon_{\perp})\epsilon_{\sigma})$  and  $r = \chi(A\epsilon_{\sigma})$ .

$$\begin{aligned}
l &= \chi(A\epsilon_{\perp})^2 + 1 = (\chi(A)^2 + 1)^2 + 1 = \chi(A)^4 + 2\chi(A)^2 + 2 \\
r &= \chi(A)^2 + 1 \\
l &> r
\end{aligned}$$

(44) Let  $l = \chi((N[x.P])[a.A, b.B])$  and  $r = \chi(N[x.P[a.A, b.B]])$ .

$$\begin{aligned}
l &= \chi(N[x.P])^2(\chi(A) + \chi(B)) + 1 \\
&= (\chi(N)^2\chi(P) + 1)^2(\chi(A) + \chi(B)) + 1 \\
&= (\chi(N)^4\chi(P)^2 + 2\chi(N)^2\chi(P) + 1)(\chi(A) + \chi(B)) + 1 \\
&= \chi(N)^4\chi(P)^2(\chi(A) + \chi(B)) + \chi(N)^2(2\chi(P)(\chi(A) + \chi(B))) \\
&\quad + \chi(A) + \chi(B) + 1 \\
r &= \chi(N)^2\chi(P[a.A, b.B]) + 1 \\
&= \chi(N)^2(\chi(P)^2(\chi(A) + \chi(B)) + 1) + 1 \\
&= \chi(N)^2\chi(P)^2(\chi(A) + \chi(B)) + \chi(N)^2 + 1 \\
l &> r
\end{aligned}$$

□

**Theorem 23.** *The calculus  $\mathbf{F}_{\forall, \exists, \rightarrow, \wedge, \vee, \perp}$  is strongly normalizing.*

*Proof.* Suppose that

$$M^{\tau} = M_0^{\tau} \rightarrow M_1^{\tau} \rightarrow M_2^{\tau} \rightarrow \dots$$

If there is infinitely many  $\beta$ -reductions in the sequence above then we have an infinite reduction in  $\mathbf{F}_{\forall, \rightarrow}$ . If almost all reduction steps are of type  $\rightsquigarrow$  then we use Lemma 22. In both cases we reach contradiction. □

## 4 Summary

We have presented a short proofs of strong normalization for simply-typed and polymorphic  $\lambda$ -calculus with all connectives. Syntax-driven translations used in those proofs allow to reduce the SN property problem to calculi with less number of connectives.

The CPS-translation used for polymorphic case looks may be helpful dealing with higher level  $\lambda$ -calculus such as  $\mathbf{F}_{\omega}$ . This is our next research problem.

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